Dynamic Fields

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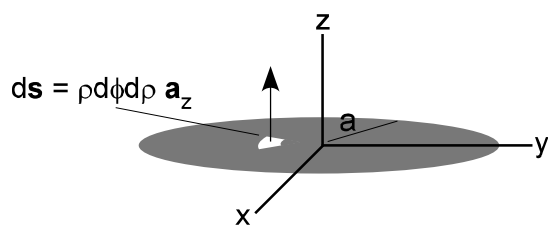
Charge Conservation

Current density, **J**, is a vector valued current per area, in SI units A/m^2. Performing a surface integral with **J** gives the total current passing through an area.

$$I = \iint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

For example, the current in the $+\mathbf{a}_z$ direction through the surface below is $I = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \mathbf{J} \bullet d\mathbf{s}$



If integration done about a **closed surface**, the result is the net current leaving the volume enclosed by the closed surface. Assuming that charge is neither created nor destroyed within the volume, the net current out of the volume must equal the time rate of change of charge stored within the volume. Here d**s** is assumed to be directed out of the volume.

$$I_{out} = -\frac{dQ}{dt} = \oiint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

Charge conservation is a generalization of KCL, where it is assumed that nodes cannot store charge.

Point form of charge conservation

Like any vector, the divergence theorem holds for current density.

$$\iiint_{\text{volume}} \nabla \bullet \mathbf{J}\, dv = \oiint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

Using charge conservation

Divergence holds for current density, which is electric charge flux.
From the definition of charge and current density:

$$I_{out} = -\frac{dQ}{dt} = \oiint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

$$-\frac{dQ}{dt} = -\frac{d}{dt}\iiint_{\text{volume}} \rho\, dv \underbrace{=}_{\substack{\text{stationary}\\ \text{volume}}} \iiint_{\text{volume}} -\frac{\partial \rho}{\partial t}\, dv = \oiint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

Where ρ is volume charge density. This result, together with the divergence theorem,

$$\iiint_{\text{volume}} \nabla \bullet \mathbf{J}\, dv = \oiint_{\text{surface}} \mathbf{J} \bullet d\mathbf{s}$$

gives the point form of charge conservation.

$$\nabla \bullet \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

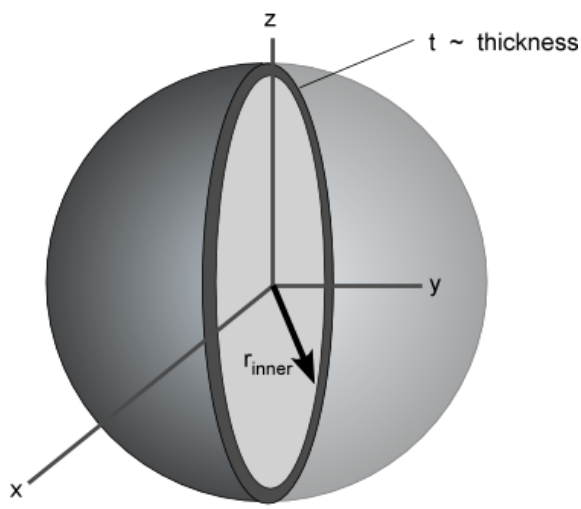
Consider the physical meaning of the divergence of **J**. The divergence of the current density must be the net current out per unit volume. The net current out per volume is the time rate of change of the net charge per unit volume leaving a point.

Example: spherical shell

The current density in a spherical shell, $\mathbf{J} = 100\ r\ \mathbf{a}_r$ A/m^2. Given $r_{inner} = 0.1$ m and t = 1 cm, answer the following questions by using surface integrals. Check your answer using volume integrals.

i) Is the charge within the spherical shell increasing or decreasing with time?

ii) At what rate is the charge increasing or decreasing?



For this problem, the surface integrals will be of the form $I_{out} = -\frac{dQ}{dt} = \oiint_{surface} \mathbf{J} \bullet d\mathbf{s}$

The volume integrals will be of the form $I_{out} = -\frac{dQ}{dt} = \iiint_{volume} \nabla \bullet \mathbf{J}\, dv$

using surface integrals

$$-\frac{dQ}{dt} = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} 100\ (0.11)\mathbf{a}_r \bullet (0.11)^2\ \sin\theta\ d\theta d\varphi \mathbf{a}_r + \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} 100\ (0.1)\mathbf{a}_r \bullet (0.1)^2\ \sin\theta\ d\theta d\varphi(-\mathbf{a}_r)$$

$$-\frac{dQ}{dt} = 100\ (0.11)^3\ 4\pi - 100\ (0.1)^3\ 4\pi$$

$$\frac{dQ}{dt} \cong -0.416 \text{ C/s}$$

check using volume integrals

To use $I_{out} = -\frac{dQ}{dt} = \iiint_{volume} \nabla \bullet \mathbf{J}\, dv$, the divergence of **J** is needed.

$$\nabla \bullet J = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(100\, r)\right] = 300 \ \ A/m^3$$

$$-\frac{dQ}{dt} = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{r=0.1}^{0.11} 300\, r^2 \sin\theta\, dr\, d\varphi\, d\theta = 300\left(\frac{0.11^3 - 0.1^3}{3}\right)4\pi$$

$$\frac{dQ}{dt} \cong -0.416 \text{ C/s}$$

The charge in the shell is decreasing at a rate of 0.416 C/s. That the charge is decreasing is reasonable since both J and the surface area increases as r increases.

More current is leaving the outer surface of the spherical shell (where r is greater) than is entering the inner surface.

Charge neutrality and relaxation time

If a net charge were introduced in a material, does the material return to a neutral state or does the material remain charged? The answer lies in the material's conductivity. In a good conductor, charge moves rapidly and a localized net charge is rapidly eliminated. On the other hand, if a net charge were introduced in a good dielectric (perhaps by ion implantation), the material would maintain a net charge for an extended period.

From charge conservation and Gauss' law,

$$\nabla \bullet \mathbf{J} = \nabla \bullet \sigma \mathbf{E} = -\frac{d\rho}{dt} \qquad\qquad \nabla \bullet \mathbf{D} = \nabla \bullet \varepsilon \mathbf{E} = \rho$$

Combining the two equations to eliminate **E**,

$$\frac{\varepsilon}{\sigma}\frac{d\rho}{dt} + \rho = 0 \qquad \rightarrow \qquad \rho(t) = \rho_o e^{-\frac{\sigma}{\varepsilon}t}$$

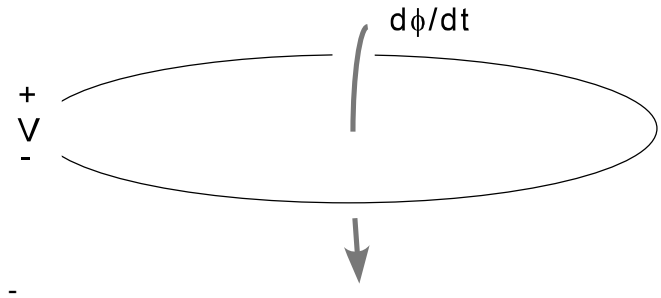
Where $\frac{\varepsilon}{\sigma} = \tau$ is the "relaxation time", the characteristic time required for material to return to charge neutrality.

- For copper, $\sigma = 5.8\ (10^8)$ S/m, $\varepsilon = 8.854(10^{-12})$ F/m, $\tau = 1.5\ (10^{-19})$ s
- For undoped silicon, $\sigma = 4.4\ (10^{-4})$ S/m, $\varepsilon = 1.045(10^{-11})$ F/m, $\tau = 2.4\ (10^{-7})$ s
- For glass, $\sigma = 10^{-12}$ S/m, $\varepsilon = 8.854(10^{-11})$ F/m, $\tau = 89$ s
- For fused quartz, $\sigma = 10^{-17}$ S/m, $\varepsilon = 3.365(10^{-11})$ F/m, $\tau = 3.4(10^6)$ s = 39 days

Faraday's law

In the early 1800's Michael Faraday discovered that when a loop of wire encircles a time-varying magnetic field, a voltage is induced about the loop of wire.

In the loop of wire shown below, a voltage will be induced if a time-varying magnetic field passes through the area defined by the loop.



How is V related to the changing flux and how does one predict its polarity? Faraday found the relation to be

$$V = -\frac{d\phi}{dt}$$

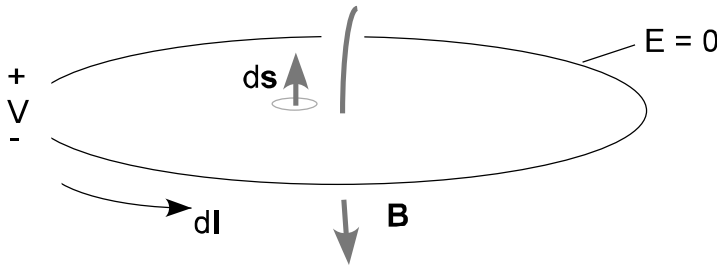
In terms of the electric field and magnetic flux density vectors,

$$\oint_{\text{loop}} \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt} \iint_{\text{surface}} \mathbf{B} \bullet d\mathbf{s}$$

d**l** and d**s** are related by the right-hand rule. With the fingers of your right hand in the direction of d**l**, your thumb will be in the direction of d**s**.

The induced voltage is equal to the path integral of the electric field, $V = \oint_{\text{loop}} \mathbf{E} \bullet d\mathbf{l}$

Consider for the moment that the wire is a perfect conductor, and begin the path integral at the negative sign of V in the direction indicated below. Putting the fingers of your right hand in the direction of d**l**, results in the thumb of your right hand being directed upward, which is the direction of d**s**.



If the path is taken within the PEC, the path integral in the conductor will yield nothing and the induced voltage is $V = \int_{+}^{-} \mathbf{E} \bullet d\mathbf{l}$. What is required for the induced voltage to be **positive**? As can be seen from the diagram d**s** is upward (this is determined by the RHR from d**l**) so that $\mathbf{B} \bullet d\mathbf{s}$ in the relation below is negative.

$$V = -\frac{d}{dt} \iint_{surface} \mathbf{B} \bullet d\mathbf{s}$$

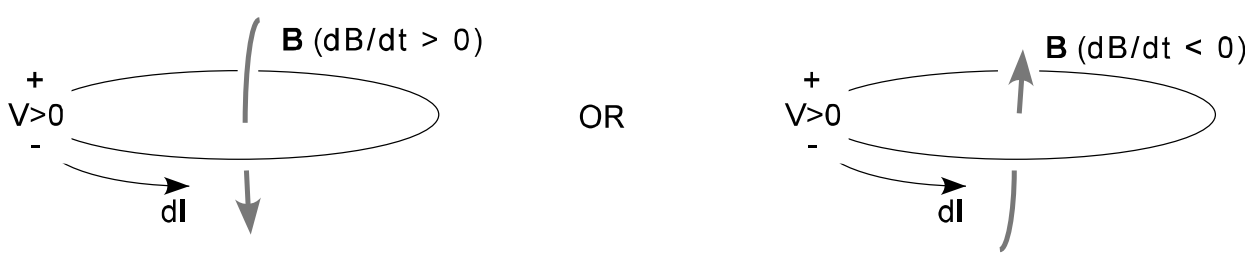
In order for V is be positive, the time derivative must be positive. So, with **B** directed as above, with d**s** and **B** in opposite directions, V will be positive <u>if</u> **B** is increasing with time.

Another arrangement would also result in V being positive. If **B** were directed upward rather than downward and were decreasing rather than increasing with time. If these conditions were met, V would again be positive. **B** and d**s** would be in the same direction so that $\mathbf{B} \bullet d\mathbf{s}$ would be positive. The only way to end with a positive result would be for the time derivative, d/dt, to introduce a negative sign – **B** must be decreasing with time in order for V to be positive.

From this discussion, one can see that, for V > 0, two possibilities exist. The first is that **B** is directed <u>downward</u> and is <u>increasing</u> with time.

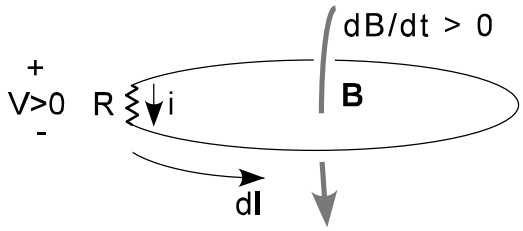
$$V = -\frac{d\phi}{dt} \qquad \rightarrow \qquad \oint_{loop} \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt} \iint_{surface} \mathbf{B} \bullet d\mathbf{s}$$

The other possibility is that **B** be directed <u>upward</u> and is <u>decreasing</u> with time.



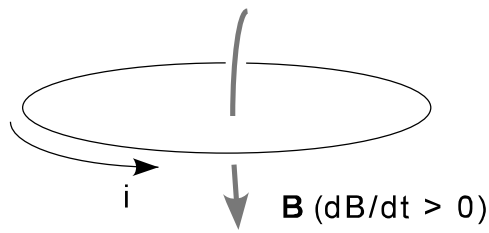
Lenz's law

We are now in a position to appreciate the significance of the negative sign in Faraday's law, referred to as Lenz's law. Connect a resistance to one of the loops above. One can see, that for V>0, the direction of current flow will be in the same direction as d**l**.



Now, eliminate the resistor and consider the wire to be a non-perfect conductor. The direction of current flow would be unchanged.

Consider this: If the rate that B were increasing would rise, i would increase. The magnetic field from i would be directed opposite to the flux which caused it to increase. There's a negative feedback mechanism at work.



Consider what would happen without Lenz's law. In this case an increase in dB/dt would result in a increase in a current whose flux would add to the flux which caused it, thereby further increasing the current and so on. This would violate the conservation of energy. It is not physical.

When Faraday began his experiments, it was not known what quantity was induced in a loop linked by a changing magnetic field. Was the induced quantity a current or a potential?

To distinguish between the two possibilities, Faraday used several rings of the same size and thickness, but made from different materials and measured the current. He found that the current was less in those rings with lower conductivity and higher in those rings made from a material with higher conductivity.

Upon further investigation, Faraday found the current was proportional to the conductivity. With this experimental evidence, he proposed that the induced quantity in a loop was an electromotive force (a voltage) and that the current present in a shorted loop is simply Ohm's law at work.

Faraday's law – transformer and motional emf

Starting with the integral form of Faraday's law,

$$V = -\frac{d\phi}{dt} \qquad \rightarrow \qquad \oint_{loop} \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt}\iint_{surface} \mathbf{B} \bullet d\mathbf{s}$$

If the surface over which the surface integral is taken is stationary in space, the only possible time variation of

$$\iint_{surface} \mathbf{B} \bullet d\mathbf{s}$$

is in the explicit time variation of **B**. In this case, the total time derivative outside the surface integral can be replaced by a partial derivative with respect to time operating on **B**.

$$V = -\iint_{surface} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} \qquad \text{transformer emf}$$

This electromotive force is termed **transformer emf** since the coils of transformers are typically stationary.

<u>Motional emf</u>

Consider a conductor moving in a magnetic field. The charges within the conductor experience the magnetic component of the Lorentz force. The force per-unit charge is an electric field (granted, it is not an electrostatic field, but it is an electric field and places a force on an electric charge).

$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B}) \qquad \rightarrow \qquad \mathbf{E}_{motional} = \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B}$$

The induced motional emf is $\oint \mathbf{E} \bullet d\mathbf{l} = \oint (\mathbf{v} \times \mathbf{B}) \bullet d\mathbf{l}$

For a combination of motion and time-variation in the **B** field, these components add to give the total expression for the **induced** voltage.

$$V = -\iint_{surface} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \oint (\mathbf{v} \times \mathbf{B}) \bullet d\mathbf{l} \qquad (d\mathbf{l} \text{ and } d\mathbf{s} \text{ related by RHR})$$

Example: motional and transformer emf

A 100 T coil moves in a magnetic field produced by a current i(t). The motion of the coil is in the x=0 plane with a velocity **v**.

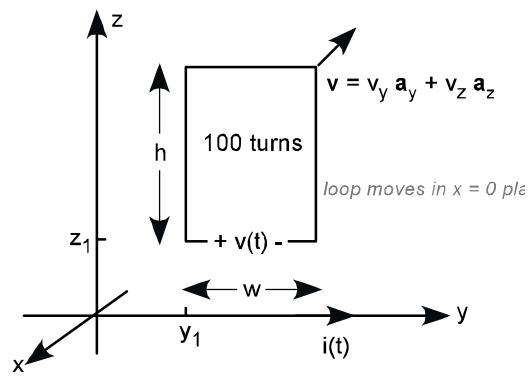
i) Given an expression for **B** in the x = 0 plane (assume air).

ii) find integral(s) for v(t) in the diagram shown below.

i) For a current on the z-axis in the $\mathbf{a}_z$ direction, the resulting magnetic flux density is

$$\mathbf{B} = \frac{\mu\, i}{2\pi\rho}\mathbf{a}_\phi$$

In this expression, ρ is the distance from the current, m is the permeability and $\mathbf{a}_\phi$ is determined from the right hand rule.



Adapting this relation to the problem at hand, z is the distance from the current, μ_o is the permeability of air, and placing your thumb along i(t) reveals a_x to be the direction of the field in the x=0 plane.

$$\mathbf{B} = \frac{\mu_o\, i}{2\pi z}\mathbf{a}_x$$

ii) $$V_{1T} = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \oint_{\text{path}} (\mathbf{v} \times \mathbf{B}) \bullet d\mathbf{l}$$

$$V_{1T} = -\int_{z=z_1+v_z t}^{z_1+h+v_z t}\int_{y=y_1+v_y t}^{y_1+w+v_y t} \frac{\mu_o}{2\pi z}\frac{di}{dt}\mathbf{a}_x \bullet dy\,dz\,\mathbf{a}_x +$$

$$\int_{z=z_1+v_z t}^{z_1+h+v_z t}\left[(v_y\mathbf{a}_y + v_z\mathbf{a}_z)\times\frac{\mu_o\, i}{2\pi z}\mathbf{a}_x\right]\bullet dz\,\mathbf{a}_z + \int_{y=y_1+w+v_y t}^{y_1+v_y t}\left[(v_y\mathbf{a}_y + v_z\mathbf{a}_z)\times\frac{\mu_o\, i}{2\pi z}\mathbf{a}_x\right]\Bigg|_{z=z_1+h+v_z t}\bullet dy\,\mathbf{a}_y$$

$$+\int_{z=z_1+h+v_z t}^{z_1+v_z t}\left[(v_y\mathbf{a}_y + v_z\mathbf{a}_z)\times\frac{\mu_o\, i}{2\pi z}\mathbf{a}_x\right]\bullet dz\,\mathbf{a}_z + \int_{y=y_1+v_y t}^{y_1+w+v_y t}\left[(v_y\mathbf{a}_y + v_z\mathbf{a}_z)\times\frac{\mu_o\, i}{2\pi z}\mathbf{a}_x\right]\Bigg|_{z=z_1+v_z t}\bullet dy\,\mathbf{a}_y$$

$$V_{1T} = -\frac{\mu_o w}{2\pi}\ln\left(\frac{z_1 + h + v_z t}{z_1 + v_z t}\right)\frac{di}{dt} \qquad \text{(transformer emf)}$$

$$- \frac{\mu_o}{2\pi} i \ln\left(\frac{z_1 + h + v_z t}{z_1 + v_z t}\right) \qquad \text{(from first part of path integral)}$$

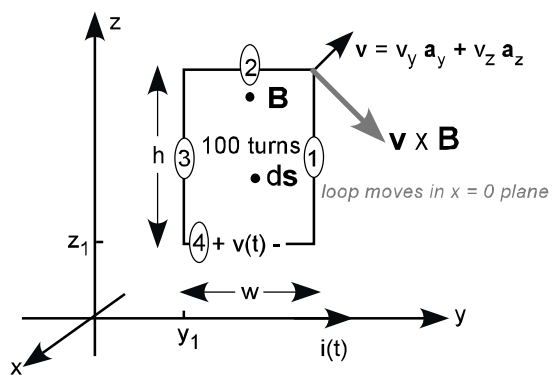
$$- \frac{\mu_o w\, v_z}{2\pi\left(z_1 + h + v_z t\right)} i \qquad \text{(from second part)}$$

$$+ \frac{\mu_o}{2\pi} i \ln\left(\frac{z_1 + h + v_z t}{z_1 + v_z t}\right) \qquad \text{(from third part)} \quad - \frac{\mu_o}{2\pi} i \ln\left(\frac{z_1 + v_z t}{z_1 + h + v_z t}\right) = \frac{\mu_o}{2\pi} i \ln\left(\frac{z_1 + h + v_z t}{z_1 + v_z t}\right)$$

$$+ \frac{\mu_o w\, v_z}{2\pi\left(z_1 + v_z t\right)} i \qquad \text{(from fourth part)}$$

$$V = 100\, V_{1T}$$

Understanding is deepened when the problem is viewed in a qualitative light. For example, given that the current is increasing with time, should we expect the transformer emf to be positive or negative?



If the current is positive, should we expect the motional emf to be positive or negative? How do the four components of the path integral contribute to the overall result for motional emf?

<u>Transformer emf</u>: From the direction of d**s** and **B**, it may be seen that the transformer emf should be negative if the field is increasing with time.

<u>Motional emf</u>: Along path 1, the first part of the path integral from the negative sign of v(t) to the positive sign, the motional emf term $\mathbf{v} \times \mathbf{B}$ will tend to push a positive charge toward the negative (-) of v(t). This contributes negatively to v(t) so that, in the sum, its contribution should appear as negative. As one can see above it does.

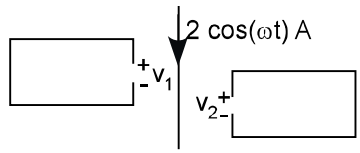
Similarly, the motional term along path 2 should appear negative in the sum, along path 3 positive, and along path 4 positive.

Contributions from 1 and 3 will be equal (they're the same distance from the current and so see the same field) and opposite and will completely cancel.

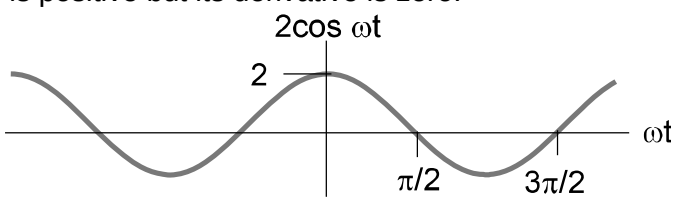
Contributions from 2 and 4 will cancel, but not completely. Path 4 will dominate since it is closer to the current and sees a stronger field.

Example: transformer and motional EMF

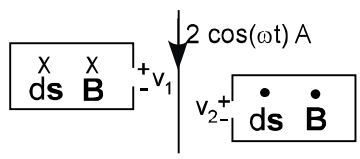
Transformer EMF: Determine whether voltages v_1 and v_2 across the stationary loops are positive, negative, or zero for i) $\omega t = 0$, ii) $\omega t = \pi/2$, and iii) $\omega t = 3\pi/2$.



i) for $\omega t = 0$, $\cos(\omega t)$ is positive but its derivative is zero.



This and the direction of **B** and d**s** provide the information needed to determine whether v(t) is positive, negative or zero.



The determination centers on the evaluation of the integral (stationary loops).

$$V = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s}$$

From the diagrams above, it can be seen that **B** and d**s** are in the same direction so that the dot product of B and ds is positive. To cancel the negative sign and obtain a positive voltage, d/dt will need to introduce another negative sign (**B** decreasing with time).

Here **B** is not changing with time. Its derivative is zero – **B** is at a maximum. In this case, therefore, V = 0.

ii) Here, **B** is decreasing with time. Following the reasoning given above, V > 0.

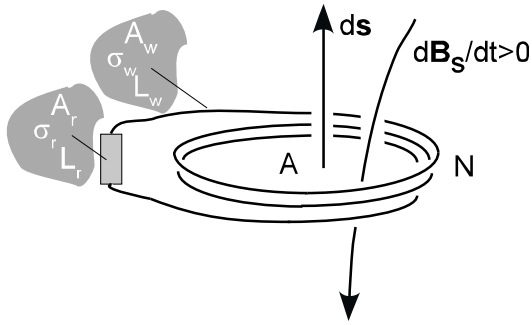
iii) Here, **B** is increasing with time. Following the reasoning given above, V < 0..

Motional EMF: Consider the loops to be moving. Assume the loop on the right is moving straight up, parallel to the current, and that the loop at the left is moving left, away from the current.

Determine whether the motional EMF for v_1 and v_2 are positive, negative, or zero for iv) $\omega t = 0$, v) $\omega t = \pi/2$, and vi) $\omega t = \pi$

Example: in-class exercise

Work with neighbors to develop a reasonable, physics-based circuit model of the system shown.



Faraday's law – transformer and motional emf terms

$$\oint_{\text{path}} \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt} \iint_{\text{surface}} \mathbf{B} \bullet d\mathbf{s}$$

For a function of time and space, the total time derivative can be split into two operators, the first a partial derivative of time for explicit time dependence and the other linked to movement and spatial changes.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla$$

Using this approach on the differential relation in Faraday's law.

$$\mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt}(\mathbf{B} \bullet d\mathbf{s}) = -\frac{\partial}{\partial t}(\mathbf{B} \bullet d\mathbf{s}) - (\mathbf{v} \bullet \nabla)(\mathbf{B} \bullet d\mathbf{s})$$

Taking this result with the vector identity below.

$$(\mathbf{v} \bullet \nabla)\mathbf{B} = \nabla \times (\mathbf{B} \times \mathbf{v}) + (\mathbf{B} \bullet \nabla)\mathbf{v} - \mathbf{B}(\nabla \bullet \mathbf{v}) + \mathbf{v}(\nabla \bullet \mathbf{B})$$

which reduces to $(\mathbf{v} \bullet \nabla)\mathbf{B} = \nabla \times (\mathbf{B} \times \mathbf{v}) + \mathbf{v}(\nabla \bullet \mathbf{B})$ for constant $\mathbf{v}$.

Using these results in the integral form of Faraday's law, we have

$$\oint_{\text{path}} \mathbf{E} \bullet d\mathbf{l} = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} - \iint_{\text{surface}} \left[\nabla \times (\mathbf{B} \times \mathbf{v}) + \mathbf{v}(\nabla \bullet \mathbf{B})\right] \bullet d\mathbf{s}$$

$$\oint_{\text{path}} \mathbf{E} \bullet d\mathbf{l} = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} - \iint_{\text{surface}} \left[\nabla \times (\mathbf{B} \times \mathbf{v})\right] \bullet d\mathbf{s} \qquad (\text{since } \nabla \bullet \mathbf{B} = 0)$$

$$\oint_{\text{path}} \mathbf{E} \bullet d\mathbf{l} = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \iint_{\text{surface}} \left[\nabla \times (\mathbf{v} \times \mathbf{B})\right] \bullet d\mathbf{s} \qquad (-\mathbf{B} \times \mathbf{v} = \mathbf{v} \times \mathbf{B})$$

$$\oint_{\text{path}} \mathbf{E} \bullet d\mathbf{l} = -\iint_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \oint_{\text{path}} (\mathbf{v} \times \mathbf{B}) \bullet d\mathbf{l} \qquad (\text{using Stoke's theorem})$$

$$\frac{\partial^2 g}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}$$

Solutions for the traveling wave g have the argument (t ± z/v), with the +sign corresponding to a wave traveling in the –$\mathbf{a}_z$ direction and the –sign corresponding to a wave traveling in the +$\mathbf{a}_z$ direction. The form of the solutions are

$$g = g\left(t \pm \frac{z}{v}\right)$$

This solution satisfies the wave equation. Assigning the argument $\alpha = t \pm z/v$, substituting and using the chain rule of differentiation, first for the spatial derivative,

$$\frac{\partial g}{\partial z} = \frac{\partial g}{\partial \boldsymbol{\alpha}}\frac{\partial \boldsymbol{\alpha}}{\partial z} = \frac{\partial g}{\partial \boldsymbol{\alpha}}\left(\pm\frac{1}{v}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial^2 g}{\partial \boldsymbol{\alpha}^2}\left(\pm\frac{1}{v}\right)^2$$

and next for the time derivative,

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \boldsymbol{\alpha}}\frac{\partial \boldsymbol{\alpha}}{\partial t} = \frac{\partial g}{\partial \boldsymbol{\alpha}}(1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial^2 g}{\partial \boldsymbol{\alpha}^2}(1)^2$$

So that the form of the solution is $g = g\left(t \pm \frac{z}{v}\right)$.

The form of the function g needs to be determined. From the wave equation, one can see the requirement is for the second derivative with respect to (w.r.t.) position be equal to the second derivative w.r.t. time. Harmonic functions (sinusoids or exponentials) will work.

Using cosines,

For a wave traveling in the +$\mathbf{a}_z$ direction, $E_x^+(z,t) = E^+\cos\left(t - \frac{z}{v}\right)$

For a wave traveling in the -$\mathbf{a}_z$ direction, $E_x^-(z,t) = E^-\cos\left(t + \frac{z}{v}\right)$

These represent electric waves. The electric vector points in the $\mathbf{a}_x$ direction and travels either in the +$\mathbf{a}_z$ or the -$\mathbf{a}_z$ direction. When one says the electric vector points only in the $\mathbf{a}_x$ direction, what is meant is that the vector has only an $\mathbf{a}_x$ component and that it is positive.

Using vector notation for the wave traveling in the +$\mathbf{a}_z$ direction.

$$\mathbf{E} = E_x^+(z,t)\,\mathbf{a}_x = E^+\cos\left(t - \frac{z}{v}\right)\mathbf{a}_x$$

For the wave traveling in the -$\mathbf{a}_z$ direction.

$$\mathbf{E} = E_x^-(z,t)\,\mathbf{a}_x = E^-\cos\left(t + \frac{z}{v}\right)\mathbf{a}_x$$

A phase, ϕ, could be added to the arguments and the solutions would still satisfy the wave equation. The arguments could also be multiplied by a constant and remain solutions to the wave equation.

This leads to discussing a common alternate form of argument. This is the form of the argument that will me used most often in our work.

$$t + \frac{z}{v} = t + \frac{z}{f\lambda}$$ where f is linear frequency and λ is the wavelength

Multiplying the argument by $2\pi f$,

$$2\pi f\left(t + \frac{z}{f\lambda}\right) = 2\pi f t + \frac{z}{\lambda/2\pi} = \omega t + \beta z$$

Where $\omega = 2\pi f$ is the angular frequency and $\beta = 2\pi/\lambda$ is the wavenumber.

Writing the argument in this form,

$$\mathbf{E} = E_x^+(z,t)\,\mathbf{a}_x = E^+\cos(\omega t - \beta z)\,\mathbf{a}_x$$ for wave traveling in +$\mathbf{a}_z$ direction

$$\mathbf{E} = E_x^-(z,t)\,\mathbf{a}_x = E^-\cos(\omega t + \beta z)\,\mathbf{a}_x$$ for wave traveling in -$\mathbf{a}_z$ direction

As before, one is free to add a phase to the arguments.

Using Cartesian coordinates, the vector wave equation for **E** can be expressed as:

$$\nabla^2\mathbf{E} = \mu\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$ ($\mathbf{a}_x$ component)

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2}$$ ($\mathbf{a}_y$ component)

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2}$$ ($\mathbf{a}_z$ component)

Comparing v in the scalar wave equation to the factor $\mu\varepsilon$ in the vector wave equation from Maxwell's equations, it is clear that

$$v = 1/\sqrt{\varepsilon\mu}$$

In vacuum or free space, $\varepsilon = \varepsilon_o$ and $\mu = \mu_o$, v = c

$$c = \sqrt{\frac{1}{\varepsilon_o\mu_o}} \cong \sqrt{\frac{1}{\left[8.854(10^{-12})\text{ F/m}\right]\left[4\pi(10^{-7})\text{ H/m}\right]}} = 2.998\ (10^8)\text{ m/s}$$

This result provided solid evidence that electricity, magnetism, and optics might well be simply aspects of a more general electromagnetic theory.

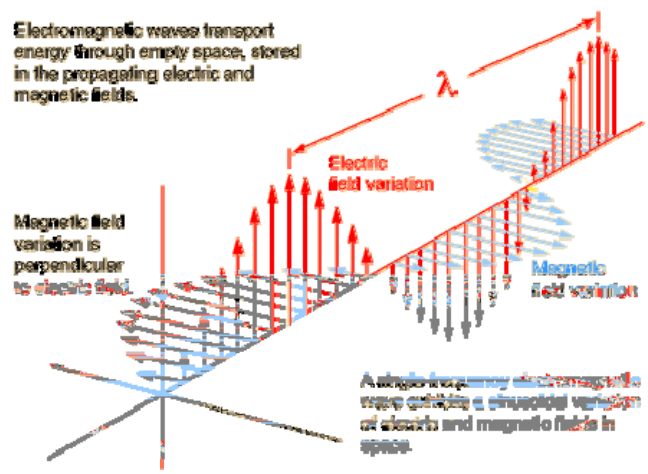
Plane waves

Plane waves are waves in which the wave fronts are planes. An example of a plane wave, consider an electric vector $\mathbf{E} = E_x\mathbf{a}_x$ and take E_x to depend only on z and t,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

The solution must be a function for which its 2nd derivative with respect to z is equal (aside from a constant multiplying factor) to its 2nd derivative with respect to t.

$$E_x^{\pm}(z,t) = E^{\pm}\cos\left(\omega t \mp \beta z + \varphi^{\pm}\right)$$



Since the wave's phase is constant across planes parallel to the x-y plane, these waves are called ***plane waves***. Since their amplitude and direction are uniform on these planes, they are called ***uniform plane waves***.

Wave travel arbitrary directions – the wave vector

The general formulation of a uniform plane wave traveling in a lossless medium includes the vector's magnitude and direction, $\mathbf{E}_o$, and describing its travel by a wave vector $\boldsymbol{\beta} = \beta\,\mathbf{a}_\beta$. The wave vector's magnitude is the wavenumber $\beta = 2\pi/\lambda$, with its direction in the direction ($\mathbf{a}_\beta$) of wave travel.

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_o\cos\left[\omega t - (\boldsymbol{\beta}\bullet\mathbf{r}) + \phi\right] \text{ (T.D.)} \qquad \tilde{\mathbf{E}}(\omega,t) = \mathbf{E}_o\ e^{j\phi}e^{-j(\boldsymbol{\beta}\bullet\mathbf{r})} \quad \text{(F.D)}$$

Where $\mathbf{E}_o$ and $\boldsymbol{\beta}$ are orthogonal for a TEM wave. Expanding the relationship using Cartesian coordinates.

$$\mathbf{E}(x,y,z,t) = \left(E_{ox}\mathbf{a}_x + E_{oy}\mathbf{a}_y + E_{oz}\mathbf{a}_z\right)\cos\left[\omega t - \left(\beta_x\mathbf{a}_x + \beta_y\mathbf{a}_y + \beta_z\mathbf{a}_z\right)\bullet\left(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z\right) + \phi\right]$$

Harmonic analysis

These plane waves could describe traveling EM waves at one frequency or they could be Fourier components of more complex traveling waves.

Why use complex valued waves? The reason is that the mathematics is often simpler, the same reasons that phasors are used in circuit analysis for sinusoidal steady-state analysis.

$$E_x^+(z,t) = E^+ \cos\left(\omega t - \beta z + \varphi^+\right) = \mathrm{Re}\left[E^+ e^{j\left(\omega t - \beta z + \varphi^+\right)}\right]$$

$$E_x^+(z,t) = \mathrm{Re}\left[E^+ e^{j\varphi^+} e^{-j\beta z} e^{j\omega t}\right] = \mathrm{Re}\left[\tilde{E}^+ e^{-j\beta z} e^{j\omega t}\right]$$

(wave traveling in $+\mathbf{a}_z$ direction)

$$E_x^-(z,t) = E^- \cos\left(\omega t + \beta z + \varphi^-\right) = \mathrm{Re}\left[E^- e^{j\varphi^-} e^{j\beta z} e^{j\omega t}\right] = \mathrm{Re}\left[\tilde{E}^- e^{j\beta z} e^{j\omega t}\right]$$

(travel in $-\mathbf{a}_z$ direction)

Harmonic analysis is similar to phasor analysis (phasor analysis is a special case). Taking φ^+ and φ^- to be zero, the frequency-domain plane waves can be identified.

$$\tilde{E}_x(z,\omega) = E^+ e^{-j\beta z} + E^- e^{j\beta z}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{f\lambda} = \frac{\omega}{v_p} = \omega\sqrt{\mu\varepsilon}$$

Maxwell's equations in the frequency-domain (harmonic analysis)

From the above discussion, it is clear that as long as the field varies harmonically with time, a corresponding complex field can be constructed, where its real part is the original field.

$$\mathbf{E}(x,y,z,t) = \mathbf{E}^+ \cos\left[\boldsymbol{\omega} t - (\beta \bullet \mathbf{r}) + \phi^+\right] = \mathrm{Re}\left\{\mathbf{E}^+ e^{-j(\beta \bullet \mathbf{r})} e^{j\left(\omega t + \phi^+\right)}\right\} = \mathrm{Re}\left\{\mathbf{E}^+ e^{-j(\beta \bullet \mathbf{r})} e^{j\phi^+} e^{j\omega t}\right\}$$

$$\mathbf{E} = \mathrm{Re}\left\{\tilde{\mathbf{E}} e^{j\omega t}\right\} \qquad \tilde{\mathbf{E}} = \mathbf{E}^+ e^{-j(\beta \bullet \mathbf{r})} e^{j\phi^+}$$

Just as in phasor analysis, the derivative with respect to time corresponds to a $j\omega$ multiplier in the frequency domain.

$$\frac{\partial}{\partial t}\mathbf{E} = \frac{\partial}{\partial t}\mathrm{Re}\left\{\tilde{\mathbf{E}} e^{j\omega t}\right\} = \mathrm{Re}\left\{\frac{\partial}{\partial t}\tilde{\mathbf{E}} e^{j\omega t}\right\} = \mathrm{Re}\left\{j\omega\tilde{\mathbf{E}} e^{j\omega t}\right\}$$

$$\frac{\partial}{\partial t}\mathbf{E} \leftrightarrow j\omega\tilde{\mathbf{E}}$$

In the frequency domain, Maxwell's equations become

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$$

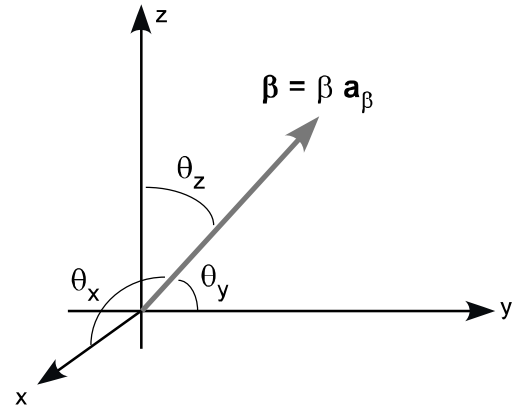
$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}$$

$$\nabla \bullet \tilde{\mathbf{D}} = \tilde{\rho}$$

$$\nabla \bullet \tilde{\mathbf{B}} = 0$$

Example: 100 MHz electric wave traveling in air, into the center of the first octant

$$\mathbf{a}_\beta = \left(\frac{1}{\sqrt{3}}\,\mathbf{a}_x + \frac{1}{\sqrt{3}}\,\mathbf{a}_y + \frac{1}{\sqrt{3}}\,\mathbf{a}_z\right)$$



Choosing **E** perpendicular to travel

$$\mathbf{a}_E = \left(\frac{1}{\sqrt{6}}\,\mathbf{a}_x + \frac{1}{\sqrt{6}}\,\mathbf{a}_y - \frac{2}{\sqrt{6}}\,\mathbf{a}_z\right)$$

Since the wave is traveling directly into the center of the first octant, and since it is clear that

$$(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2 = 1$$

The angles for the directional cosines, all equal are,

$$\cos^{-1}\left(1/\sqrt{3}\right) \cong 54.7°$$

From the propagation speed and the frequency, the wavelength and wavenumber can be found,

$$\lambda = \frac{c}{f} = \frac{3\left(10^8\right)\ \text{m/s}}{100\left(10^6\right)\ \text{Hz}} = 3\ \text{m} \qquad \beta = 2\pi/3\ \text{m}^{-1}.$$

$$\boldsymbol{\beta} = \beta\,\mathbf{a}_\beta = \beta\left(\cos\theta_x\mathbf{a}_x + \cos\theta_y\mathbf{a}_y + \cos\theta_z\mathbf{a}_z\right)$$

Suppose further that the amplitude of the electric wave is 100 V/m. Given that **E** and **β** are perpendicular, it is required that

Therefore, the traveling wave is described in the time domain as

$$\mathbf{E} = \left(\frac{1}{\sqrt{6}}\,\mathbf{a}_x + \frac{1}{\sqrt{6}}\,\mathbf{a}_y - \frac{2}{\sqrt{6}}\,\mathbf{a}_z\right) 100\cos\left(2\pi 10^8\ t - \frac{x}{\sqrt{3}} - \frac{y}{\sqrt{3}} - \frac{z}{\sqrt{3}}\right)\ \text{V/m}$$

In the frequency domain, the wave is described as

$$\tilde{\mathbf{E}} = \left(\frac{1}{\sqrt{6}}\,\mathbf{a}_x + \frac{1}{\sqrt{6}}\,\mathbf{a}_y - \frac{2}{\sqrt{6}}\,\mathbf{a}_z\right) 100\, e^{-j\left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}}\right)}\ \text{V/m}$$

TEM waves

Consider a frequency-domain description of a time-harmonic uniform plane wave, pointing in the $\mathbf{a}_x$ direction, and traveling in the $\mathbf{a}_z$ direction.

$$\tilde{\mathbf{E}} = \mathbf{a}_x \tilde{E} e^{-j\beta z}$$

The accompanying magnetic vector can be found from Faraday.

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$

Solution

$$\tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}} = \frac{1}{-j\omega} \nabla \times \tilde{\mathbf{E}} \qquad \rightarrow \qquad \tilde{\mathbf{H}} = \frac{j}{\mu\omega} \nabla \times \tilde{\mathbf{E}}$$

$$\tilde{\mathbf{H}} = \frac{j}{\mu\omega}\left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{a}_z\right]$$

$$\tilde{\mathbf{H}} = \frac{j}{\mu\omega}\left[\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial \tilde{E} e^{-j\beta z}}{\partial z} - \frac{\partial 0}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial 0}{\partial x} - \frac{\partial \tilde{E} e^{-j\beta z}}{\partial y}\right)\mathbf{a}_z\right]$$

$$\tilde{\mathbf{H}} = \frac{j}{\mu\omega}\left(-j\beta \tilde{E} e^{-j\beta z}\right)\mathbf{a}_y = \frac{\beta}{\mu\omega}\tilde{E} e^{-j\beta z}\,\mathbf{a}_y = \frac{\omega\sqrt{\varepsilon\mu}}{\mu\omega}\tilde{E} e^{-j\beta z}\,\mathbf{a}_y = \frac{\tilde{E}}{\sqrt{\mu/\varepsilon}} e^{-j\beta z}\,\mathbf{a}_y$$

The electric and magnetic vectors perpendicular and both are perpendicular to the direction of propagation. $\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_\beta$

$\eta = \dfrac{E}{H} = \sqrt{\dfrac{\mu}{\varepsilon}}$ has units of Ω and is called the wave (or intrinsic) impedance, $\eta_o = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} \cong 377\Omega$

The result is

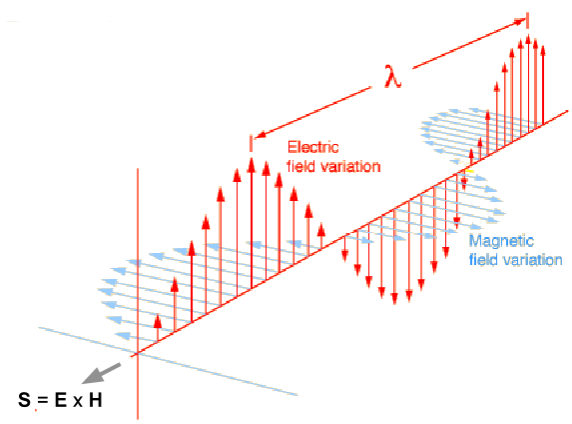
$\mathbf{H} = \mathbf{a}_\beta \times \dfrac{1}{\eta}\mathbf{E}$ or use Faraday's law

$\mathbf{E} = \eta \mathbf{H} \times \mathbf{a}_\beta$ or use Ampere's law

$\eta = \sqrt{\dfrac{\mu}{\varepsilon}} \qquad \beta = \dfrac{2\pi}{\lambda}$

$v = f\lambda = \dfrac{\omega}{\beta}$ (general)

$v = \dfrac{1}{\sqrt{\varepsilon\mu}}$ (lossless only)



Example: plane waves

Take $\varepsilon_r = 4$, $\mu_r = 1$, find i) v, ii) λ at f = 1 GHz, iii) β, iv) description of the electric vector and v) description of the magnetic vector.

i) $$v = \sqrt{\frac{1}{4\varepsilon_o\mu_o}} \cong \frac{1}{2}\sqrt{\frac{1}{\left[8.854(10^{-12})\text{ F/m}\right]\left[4\pi(10^{-7})\text{ H/m}\right]}}$$

$$v \cong 1.5\,(10^8)\text{ m/s}$$

ii) the wavelength for f = 1 GHz = 10^9 Hz

$$\lambda = \frac{v}{f} = \frac{1.5\,(10^8)\text{ m/s}}{10^9\text{ Hz}} = 0.15\text{ m}$$

iii) β for f = 1 GHz

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.15\text{ m}} = \frac{40\pi}{3}\text{ m}^{-1}$$

iv) the mathematical description for this electric field, traveling in the $+\mathbf{a}_z$ direction. The field points in the $+\mathbf{a}_x$ direction and has an amplitude of 2 mV/m. Take the phase, φ^+, to be zero.

-express in the time-domain

$$\mathbf{E} = 0.002\cos\left(2\pi 10^9\,t - \frac{40\pi}{3}z\right)\mathbf{a}_x\ \text{V/m}$$

-express as a frequency-domain vector

$$\tilde{\mathbf{E}} = 0.002\,e^{-j\frac{40\pi}{3}z}\mathbf{a}_x\ \text{V/m}$$

v) give the corresponding magnetic vector

$$\mathbf{H} = \frac{0.002}{188.5}\cos\left(2\pi 10^9\,t - \frac{40\pi}{3}z\right)\mathbf{a}_y\ \text{A/m}$$

-express as a frequency domain vector

$$\tilde{\mathbf{H}} = \frac{0.002}{188.5}\,e^{j\frac{40\pi}{3}z}\,\mathbf{a}_y\ \text{A/m}$$

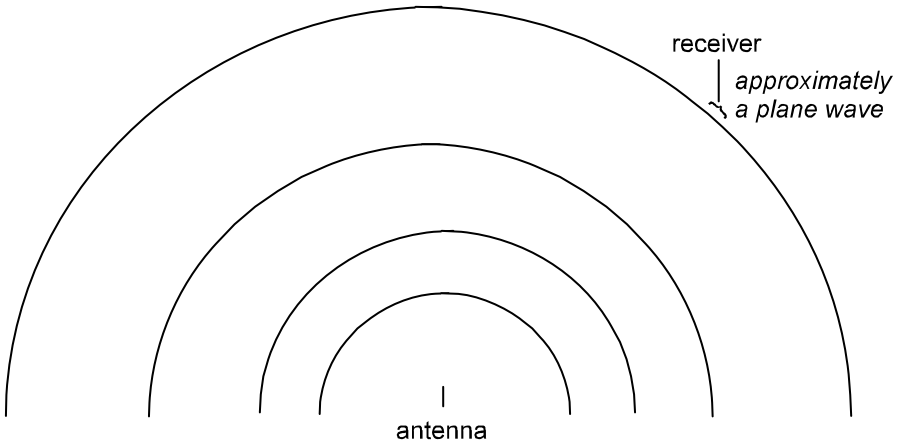
Sources for TEM waves

Strictly speaking, uniform plane waves do not exist. By definition, a plane wave extends over an entire plane and would therefore have infinite energy.

The two sources for uniform plane waves would be 1) planes of charge, varying sinusoidally in time and 2) planes of current, again varying sinusoidally in time.

Why do we study uniform plane waves when they don't even exist?

i) wave fronts that are distant from their source approximate plane waves



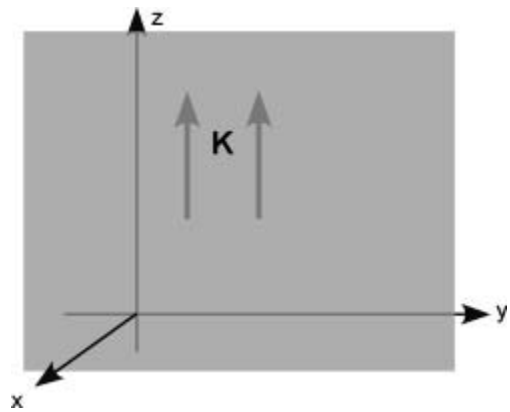
ii) using superposition, plane waves can be combined to produce more complex forms

same strategy as that of time-domain analysis [*finding system response to impulse, then using superposition (convolution) for find system response to general input]*

same strategy as in frequency-domain analysis *[finding sinusoidal response, they using superposition (Fourier) to find system response to general input]*

Example: generation of plane waves

Given the surface current $\mathbf{K} = \mathbf{a}_z 2\sin(\omega t)$ A/m on x=0 plane. Given the time-domain and frequency-domain waves for all space (assume free space or air).



Given this symmetry, it is clear that the magnetic vectors must point in $\pm\mathbf{a}_y$ direction

Harmonic solutions satisfy the wave equations obtained from Maxwell's equations. The solution will also meet boundary conditions and satisfy symmetry requirements.

From symmetry, there are two waves, one traveling in the $+\mathbf{a}_x$ direction for x>0 and the other traveling in the $-\mathbf{a}_x$ direction for x<0.

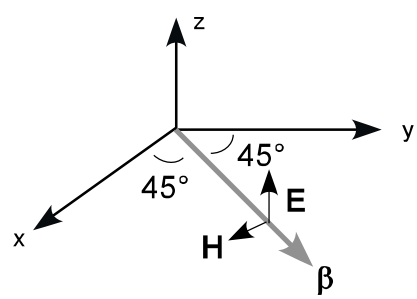
$$\text{Solutions are } \mathbf{H} = \begin{cases} \sin(\omega t - \beta x)\ \mathbf{a}_y & x>0 \\ -\sin(\omega t + \beta x)\ \mathbf{a}_y & x<0 \end{cases}.$$

To check these results, confirm that the boundary conditions for **H** are met.

$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$$

Example: transverse electromagnetic (TEM) waves

An electric field with a magnitude of 188.5 V/m points in the $\mathbf{a}_z$ direction and travels parallel to the x-y plane as shown. The wavelength is 700 nm and the medium is air. Give the frequency domain description of **E** and **H**.



$$\beta = 2\pi/700\left(10^{-9}\right)\ \text{m}^{-1} \cong 9.0\left(10^{6}\right)\ \text{m}^{-1}$$

$$\boldsymbol{\beta} = \beta\mathbf{a}_\beta = 9.0\left(10^{6}\right)\ \text{m}^{-1}\left(\cos 45°\ \mathbf{a}_x + \cos 45°\ \mathbf{a}_y\right) = 6.3\left(10^{6}\right)\mathbf{a}_x + 6.3\left(10^{6}\right)\mathbf{a}_y\ \text{m}^{-1}$$

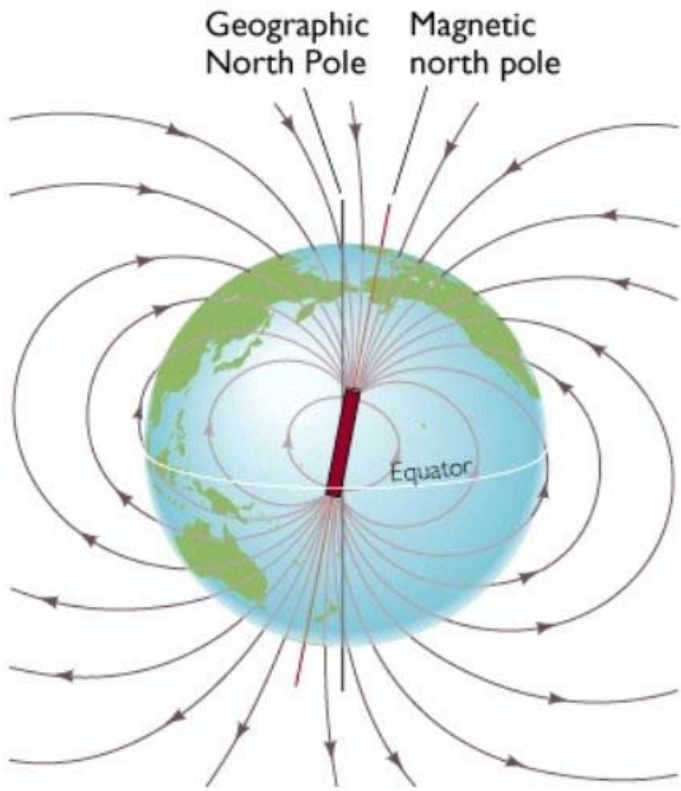
$$\mathbf{E} = 188.5\ e^{-j\left[6.3\left(10^{6}\right)x + 6.3\left(10^{6}\right)y\right]}\ \mathbf{a}_z\ \text{V/m}$$

$$\mathbf{H} = \frac{188.5}{377}\ e^{-j\left[6.3\left(10^{6}\right)x + 6.3\left(10^{6}\right)y\right]}\left(\frac{\mathbf{a}_x}{\sqrt{2}} - \frac{\mathbf{a}_y}{\sqrt{2}}\right)\ \text{A/m}$$

Earth's magnetic field

What is the source of earth's magnetic field? From the structure of the earth—with its solid, mainly iron inner core, liquid nickel-iron outer core, , plastic mantle and rocky outer crust—it is tempting to suppose that the inner and outer cores rotate at different speeds than the mantle and crust and that the core would therefore as a rotating permanent magnetic since both nickel and iron are ferromagnetic.

The problem with this conjecture is that the core is at too high a temperature for either nickel or iron to act as permanent magnets. It is characteristic of ferromagnetic materials that they have a maximum temperature, the Curie temperature (T_c), above which they cannot maintain their ferromagnetic properties. The Curie temperatures for iron and nickel are below 800 °C (360 °C for nickel and 770 °C for iron) while the outer core is at 5000 to 6000 °C and the inner core is even hotter.



Current evidence points to the earth's rotation as being the ultimate source of the fields but, rather than the core being a rotating permanent magnet, the rotating core carries electrical current which produces the magnetic field. The picture is that of a current loop giving rise to a magnetic field.

An important test case for this model is Venus, which rotates very slowly (a Venutian day is about 240 earth days). Our model would predict little magnetic field since the source of the movement carrying the electric current is largely absent in the case of Venus. This is precisely what we find.

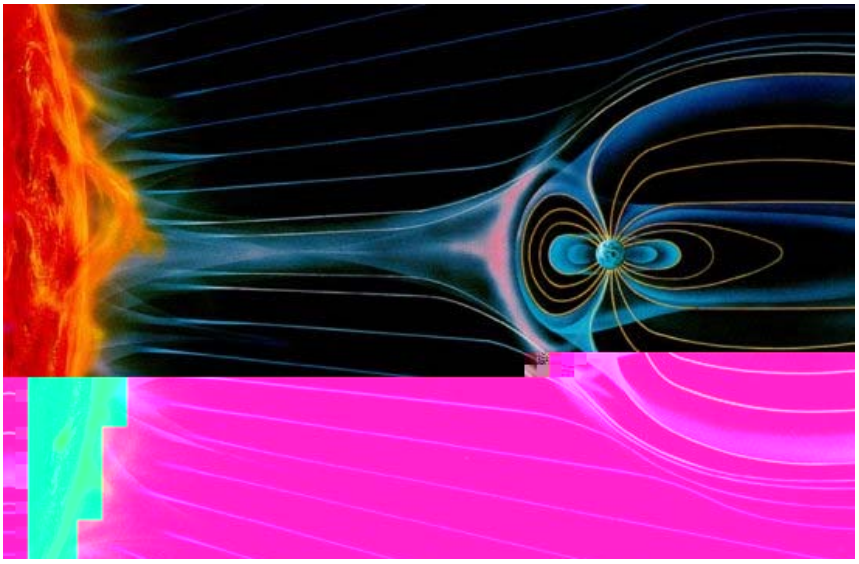
1) The geographic poles and the magnetic poles are not aligned. Moreover, geographic data provide evidence that the position of the magnetic poles vary and may flip.

2) Labeling for the magnetic poles does not follow the normal convention where the north pole is the source of flux and the south pole is the flux sink. The earth's magnetic "south pole" is the source of flux and the magnetic "north pole" is the flux sink. This results in the north pole of a compass needle point towards the earth's magnetic "north pole" which would actually be its magnetic south pole if the normal convention were followed.

3) The surface strength of earth‘s magnetic field ranges between 0.3 and 0.6 gauss (30 to 60 μT). Compare this to a typically values of 2000 gauss for flexible permanent magnets, to 3500 gauss for ceramic magnets, to 10,000 gauss for AlNiCo magnets, to over 10,000 for neodymium magnets. Also compare to the approximately 10 μT field 2 cm from a 1A line of current in air.

Aside from allowing the compass to be used for navigation, what does the earth’s magnetic field do for us? Answer: Much.

Solar wind consists of high energy charged particles (electrons, protons and ionized helium) which escape from the sun and travels radially from the sun. Vital to all surface life on earth is the fact that earth’s magnetic field deflects the solar wind which would sweep away its atmosphere.

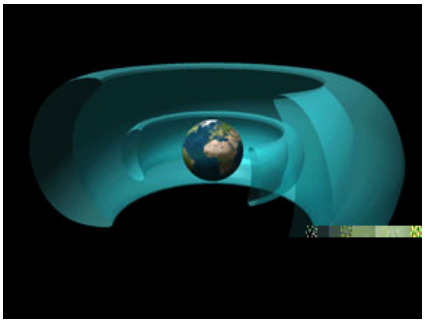


Artist rendition: National Geophysical Data Center

The **aurora borealis** (northern lights) and the aurora australis are due to energetic charged particles—trapped from the solar wind while some may be due to ionization of neutral species by cosmic rays. When these particles collide with oxygen or nitrogen, they can cause excitation to a higher energy state which, upon decay, results in photon emission.

From Northwest Suburban Astronomers

Why do the lights appear mainly at the northern and southern poles? The ionized particles are trapped by the earth's magnetosphere where they form belts of plasma—the **Van Allen radiation belts**. At the earth's magnetic poles, many of the particles are reflected, with only the most energetic traveling on the atmosphere, into the region of the poles where they collide with O_2 or N_2 which, in turn leads to the aurora display.



From University of Michigan

Comets (Greek for "hairy star") are objects which orbit about the sun, sometimes in quite elongated elliptical orbits. When far from the sun, only their heads are visible. As they draw closer to the sun, they develop two tails, an "ion tail" and a "dust tail."

The **ion tail** is composed of neutral gas molecules which are ionized by the UV photons and then swept away from the sun by solar wind and which are visible because they fluoresce (they often appear blue since the dominate ion species CO^+ scatters blue well). Being molecular in size these ions form a tail that points directly away from the sun, even when the comet moves away from the sun, and is relatively narrow.

The **dust tail** consists of more massive dust particles which have been moved away from the comet's center by radiation pressure. They are visible because they reflect the sun's light. Radiation pressure is due to momentum transfer from the photons of the sun's radiation.

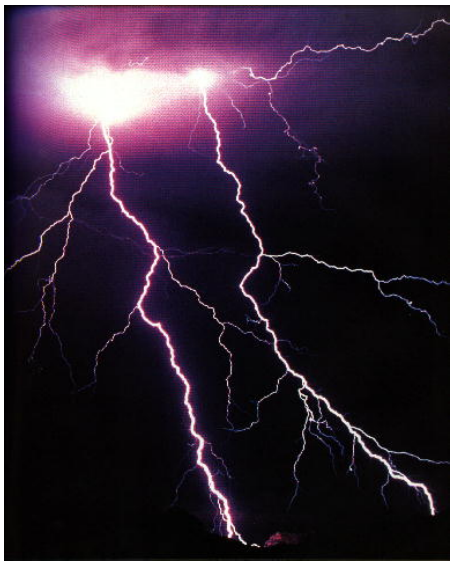
Two tails of Hale-Bopp ~ David Jewitt
Institute for Astronomy – U. of Hawaii

$$p = E/c = hf/c = h\lambda$$

An interesting detail of the dust tail is that it is slightly curved, an effect especially noticeable near the sun. This is due to the dust particles having a greater radius from the sun without having gain any speed, making their period of revolution about the sun longer.

Lightning

Raindrops experience a net loss of electrons during their trip from the clouds to the earth. The result is a tendency for the earth to be net positive. Another result is that the upper regions of clouds tend to have an excess of electrons and their lower regions to be positive. This charge separation gives rise to lightning discharges within and between clouds as well as from cloud to ground.



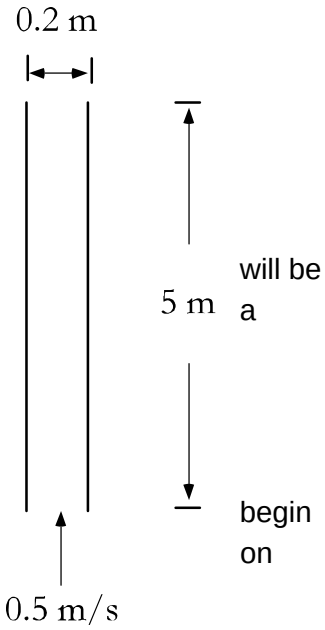
From the Franklin Institute

Charge separation between the cloud and earth produce a strong electric field. The electric field which results can be sufficiently strong to ionize the air which results in stepped leaders, negative discharges which head toward the ground. Lightning is typically initiated in the cloud since thinner air is more easily ionized. As these negative leaders approach the surface of the earth, a positive leader from the earth is formed, which upon connecting with the negative leader from the cloud, forms a complete conducting path from the cloud to ground. After the discharge path has been established the much brighter "return stroke" occurs carrying high currents and dissipating a large amount of energy.

Electrostatic filters

The emissions from coal-fired plants include a wide variety of trace materials in addition carbon compounds, including heavy metals and radioactive materials.

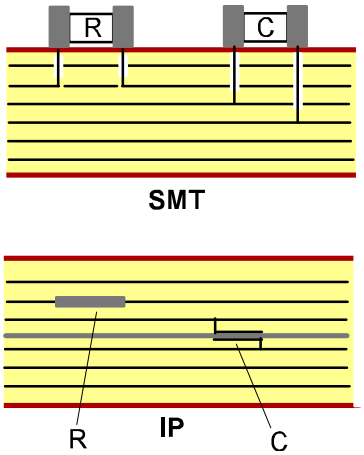
Design an electrostatic filter for an electric utility in order to remove particulate emissions from their coal-fired plants. The initial design will be an array of plate structures like that shown below. The particles have a mass of less than 1 mg and are given a charge of 10-12 C. The emissions total 2 tons/day.



Find the voltage and power required to remove the particulate emissions. To find the voltage required, allow that the particles can begin at the positive plate. To determine the power required, assume that, on average, the particles begin middle distance between the plates.

Integrated passive devices

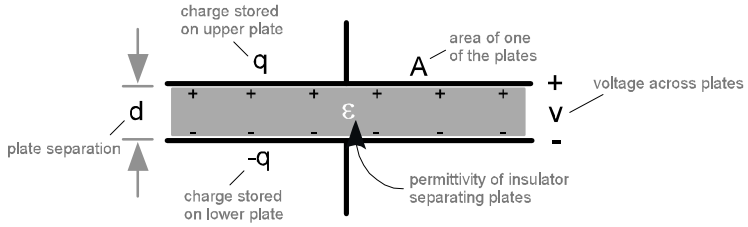
Supplying the highest frequency currents in high-speed systems is the most difficult challenge. High-frequency decoupling capacitors are placed close as possible to integrated circuit (IC) chips in order to limit inductance and careful attention is paid to their mounting. Decoupling capacitors are designed for low equivalent series resistance (ESR) and low equivalent series inductance (ESL). The purpose of this careful design and manufacture is to allow decoupling capacitors to effectively supply high-frequency current while maintaining a constant voltage. Decoupling above 100MHz present challenges when standard SMT capacitors are used and decoupling above 500MHz typically requires special low-inductance capacitors. These capacitors are often four-lead capacitors to reduce magnetic coupling which can, in turn, result in a more stable voltage at the chip. Integrated capacitors are readily designed with a lower inductance than discrete capacitors because of the fundamental different current paths. In the leads of the SMT capacitor, the current are in the same direction whereas in the plates of an integrated capacitor, the leads are much shorter and the current on the plates are in opposite directions. This is the fundamental physical reason that integrated capacitors have a much greater potential for power bus decoupling at high frequencies.



Further investigation in this area is needed for decoupling capacitors, for RC terminations for high-speed lines, and for filtering applications. Motivations for widespread use of integrated passives include size reductions through fewer mounted components, cost savings through simpler packaging, reliability improvements through fewer solders, SI improvements through shorter paths with fewer discontinuities, and high frequency performance improvements through improved placement.

Sensors

The capacitance of the structure is determined by the geometry of the conductors and the permittivity of the insulation.



Parallel Plate Capacitor

If the plate separation is small compared to their area, the capacitance is simply related to d, A, and ε.

$$C = \varepsilon \frac{A}{d}$$

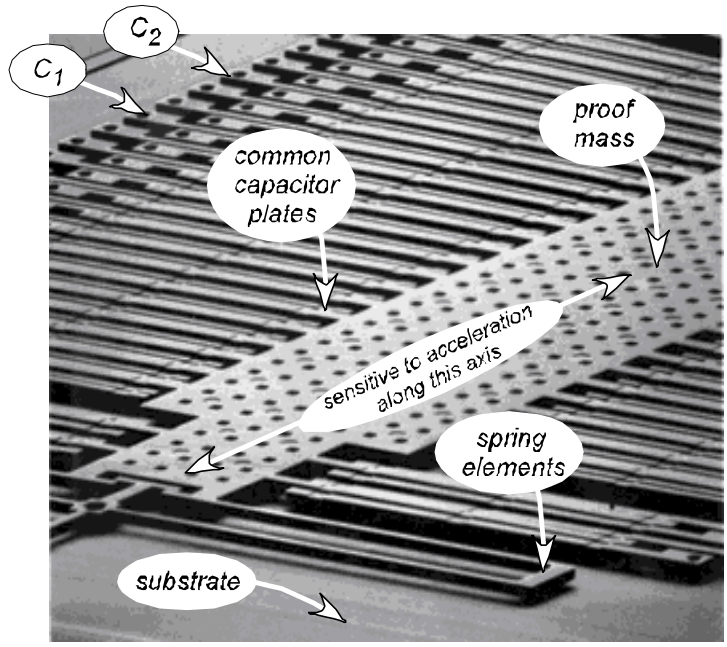
While this relation holds in detail only for the parallel plate capacitor, it does show the general trends that hold true for all capacitors. The capacitance increases as the insulator permittivity increases, increases as the conductor area increases, and decreases as the conductor separation increases.

$$\text{In summary} \quad \begin{cases} \uparrow \varepsilon, \ \uparrow A, \ \downarrow d & \Rightarrow \quad \uparrow C \\ \downarrow \varepsilon, \ \downarrow A, \ \uparrow d & \Rightarrow \quad \downarrow C \end{cases}$$

Capacitive sensors change their capacitance due to either changes in geometry (plate separation, plate area) or/and due to changes in permittivity.

Micro-electromechanical systems (MEMS) based accelerometers are standard sensors used in deploying air bags in automobiles.

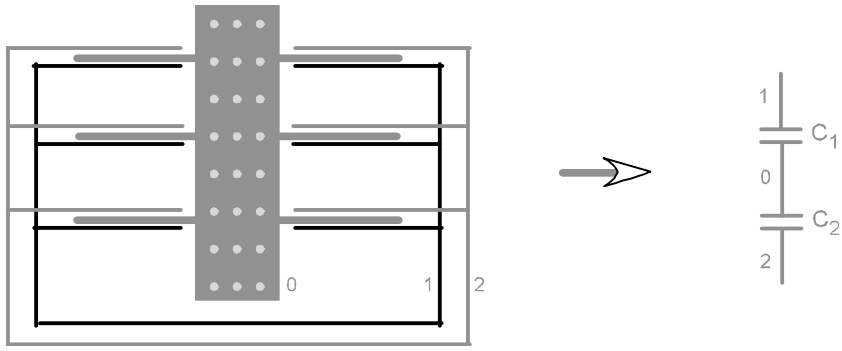
The accelerometer can be modeled as a 2nd-order system consisting of a mass, a spring, and a damper. The mass is primarily that of the proof mass.



MEMS Accelerometer
(adapted from Analog Devices)

The spring constant is determined by the spring elements between the proof mass and the accelerometer substrate. Damping is mainly determined by the viscous damping of the accelerometer atmosphere—often "squeezed film damping" due to the small dimensions of the device.

The sensor is comprised of two capacitors connected by a common plate as shown below in a simplified diagram.

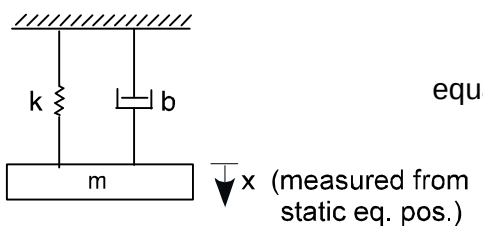


Simplified Functional Diagram of MEMS Accelerometer

The capacitors in a typical MEMS accelerometer at rest have capacitances of about 0.1 pF with a gap distance of approximately 1.3 μm.

$$C_{(1,2)} = \frac{0.1\text{ pF}}{1 \pm x/x_o} \cong 0.1\text{ pF}\left(1 \mp x/x_o\right) \qquad (\text{for } x << x_o)$$

MEMS accelerometers are often designed so that they remain very nearly in quasi-static steady-state during acceleration. That is, their time constants are much smaller than those involved in the acceleration.



equation of motion: $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Mechanics of MEMS Accelerometer

In this quasi-static case, the derivatives can be neglected.

$$kx = ma \quad \Rightarrow \quad x = \frac{ma}{k} = \frac{a}{k/m} = \frac{a}{\omega_n^2}$$

The basic trade-off can be seen to be one between sensitivity and speed. To make the device able to respond more quickly, the natural frequency should be increased—either by increasing the spring constant or by reducing the proof mass. Doing so, however will cause x to be smaller for a given acceleration and thus cause the capacitance to deviate less from its nominal value. Suppose the natural frequency, ω_n, is 50 kr/s.

What is the displacement for a = 30g?

$$x = \frac{30\ \left(9.8\ \text{m/s}^2\right)}{\left(50000\ \text{r/s}\right)^2} = 0.118\ \mu\text{m}$$

This would result in the two capacitances being

$$C_{(1,2)} = \frac{0.1\ \text{pF}}{1 \pm {}^{0.118}\!/_{1.3}} = \begin{cases} 0.092\ \text{pF} \\ 0.11\ \text{pF} \end{cases}$$

Capacitive humidity sensors employ a polymer or oxide film as the dielectric between the capacitor plates. One of the plates is porous so that the film is exposed to the atmosphere in which the humidity is being measured, and, as the humidity changes, the water absorbed by the film varies.

Capacitive Humidity Sensor (from Rotronic-USA)

Since the low frequency permittivity of water is approximately $80\varepsilon_o$ compared to around 3-10 for many polymers and oxides, it can readily be appreciated that the permittivity of the film will be dependent on the amount of water absorbed by the film, the result is that the sensor's capacitance responds to changes in humidity.

$$C = \varepsilon\frac{A}{d} \quad \Rightarrow \quad \frac{dC}{dh} = \frac{d}{dh}\left(\varepsilon\frac{A}{d}\right) = \frac{dC}{d\varepsilon}\frac{d\varepsilon}{dh} = \frac{A}{d}\frac{d\varepsilon}{dh}$$

What is required for this type of sensor is the relation between the permittivity, ε, and the humidity, h. Typical capacitances for existing sensors are between 100 and 500 pF at 50% relative humidity (RH) and 25 °C. Sensitivity is around 0.2 to 0.5 pF per 1% change in RH. Response times for capacitive humidity sensors are in the range of tens of seconds.

Table 1: Capacitive sensors

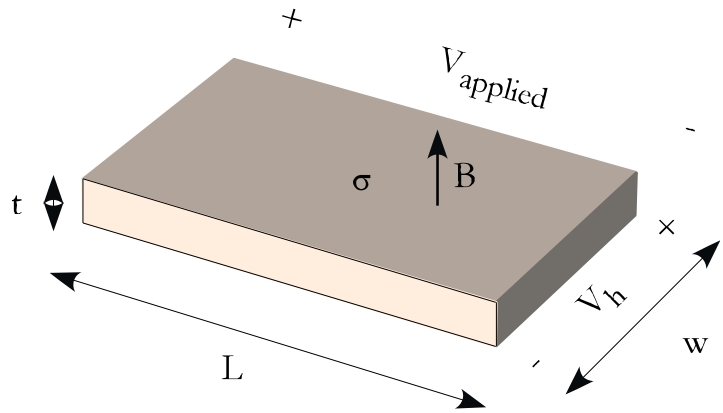
Types	Principle of Operation	Measurement Uses
Pressure	Geometry varies with pressure	Capacitive pressure sensors are typically formed by forming closed cavities which contain capacitor conductors—typically one located on a membrane exposed to pressure to be measured. Varying the pressure varies the plate separation and so varies the capacitance.
Displacement	Geometry varies with displacement	One plate is usually fixed with the other movable. Capacitance displacement sensors are made which have resolutions of less than 10 pm
Humidity	Permittivity varies with humidity	Water is absorbed by an oxide or polymer dielectric. The permittivity of the insulator between capacitor conductors depends on the amount of water absorbed which varies with humidity. Capacitive humidity sensors have become more commonly used than resistive sensors and can be used in many chemically harsh environments.
Accelerometer	Geometry varies with acceleration	Vibration control in hard-disk drives. Vibration detection in various consumer products. Avionic controls and safety systems. MEMS-based capacitive accelerometers are widely used to deploy automobile air bags.
Proximity	Geometry and/or permittivity varies with proximity	Detection of liquid level, products in inspection lines and in assembly lines.

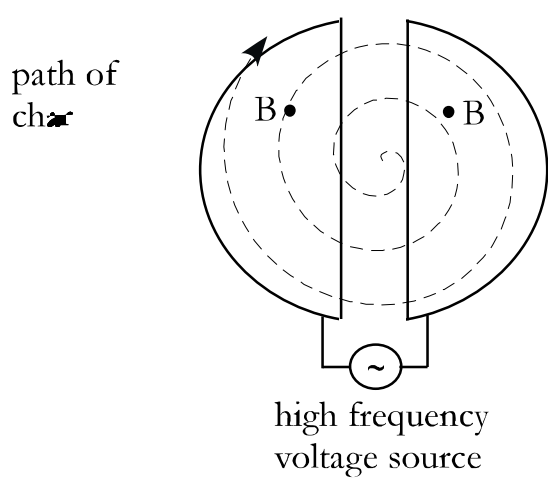
Semiconductor characterization – Hall effect

Given the doped silicon below, find V_h if the silicon is

i) doped with n phosphorous atoms per unit volume

ii) doped with p boron atoms per unit volume



High energy physics – the cyclotron

A deuterium particle with mass 3.34 (10^{-27}) kg and charge 1.6(10^{-19}) C is produced and accelerated in a cyclotron with a radius of 1m and a magnetic field of 2 Wb/m^2. Calculate the energy of the particle at the exit.